Multiparticle entanglement and its experimental detection

W Dür and J I Cirac

Institut für Theoretische Physik, Universität Innsbruck, A-6020 Innsbruck, Austria

Abstract
We discuss several aspects of multiparticle mixed-state entanglement and its
experimental detection. First we consider entanglement between two particles
which is robust against disposals of other particles. To completely detect
these kinds of entanglement, full knowledge of the multiparticle density
matrix (or of all reduced density matrices) is required. Then we review the
relation of the separability properties of l-partite splittings of a state ρ to its
multipartite entanglement properties. We show that it suffices to determine the
diagonal matrix elements of ρ in a certain basis in order to detect multiparticle
entanglement properties of ρ. We apply these observations to analyse two
recent experiments, where multiparticle entangled states of 3 (4) particles were
produced. Finally, we focus on bound entangled states (non-separable, non-
distillable states) and show that they can be activated by joint actions of the
parties. We also provide several examples which show the activation of bound
entanglement with bound entanglement.

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1. Introduction

Entanglement is at the heart of quantum information theory. In recent years, there has been
an ongoing effort to characterize entanglement quantitatively and qualitatively. While for
bipartite systems essential parts of this problem are solved, many questions remain still open
for multipartite systems. In this case, there exist several possible approaches to identify
different kinds of multiparticle entanglement (MPE), and many interesting phenomena related
to MPE have been discovered [1–6].

In this work, we review some possible approaches to identify different kinds of MPE and
discuss its experimental detection.

1.1. Bipartite entanglement, separability and distillability

Let us start with the simplest case of bipartite systems and review some basic concepts related
to bipartite entanglement. Let A and B be two spatially separated systems of dimension d_A

(d_B) respectively. A state ρ is said to be separable if it can be written as a convex combination of product states, i.e.

$$\rho = \sum_i p_i |a_i\rangle_A \langle a_i| \otimes |b_i\rangle_B \langle b_i|. \tag{1}$$

In the case where this is not possible, ρ is said to be entangled. Note that separable states ρ are states which can be prepared locally by the parties, i.e. ρ is only classically correlated. As inseparable (entangled) states are very interesting, both from a fundamental and from a practical point of view, one of the main problems in quantum information theory is the problem of establishing whether a given state ρ is separable or not. We have that condition (1) is in general very difficult to check, as there exist (in general) infinitely many ways to write a given density operator ρ as a convex combination of (possible entangled) pure states. However, the problem of separability has been extensively studied in recent years [7], and in the case of two qubits ($d_A = d_B = 2$), necessary and sufficient conditions for separability have been obtained [8, 9]. In particular, for two qubits one can use the partial transposition criterion [8, 9] which states that (i) ρ is separable iff $\rho^{T_A} \geqslant 0$ [9]. Here, T_A denotes transposition in A in a given orthonormal basis $S_A = \{|k\rangle\}_{k=1}^{d_A}$, and $X \geqslant 0$ means that all eigenvalues of X are $\geqslant 0$. For higher-dimensional systems ($d_A, d_B \geqslant 2$), positivity of the partial transposition is only a necessary, but not sufficient condition for separability.

For inseparable (entangled) density operators ρ, one may also ask whether the entanglement contained in ρ can be distilled. That is, whether out of (arbitrarily) many copies of ρ, a maximally entangled state (MES) such as the singlet state $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ shared by the parties A and B can be created by means of local operations and classical communication. In the case where this is possible, ρ is said to be distillable. Again, for two qubits it turns out that the partial transposition provides a necessary and sufficient condition for distillability: (ii) ρ is distillable iff $\rho^{T_A} \not\geqslant 0$ [10]. For higher-dimensional systems (d_A, $d_B > 2$), non-positive partial transposition is a necessary, but not sufficient condition for distillability.

The partial transposition criteria (i) and (ii) thus provide a necessary and sufficient condition for separability and distillability for two qubit systems.

1.2. Multiparticle entanglement

The aim of this paper is to extend these ideas to multiparticle systems, in particular to study separability and distillability properties of multiparticle systems. However, there are various aspects of multiparticle entanglement. For example, there exist obviously many different kinds of entanglement in a multiparticle system, as one may have bipartite entanglement shared by, say, parties A_1 and A_2 as well as bipartite entanglement shared by two other parties, say A_2 and A_3. In addition, there exist true N-partite entanglement, for example MES of N particles such as the Greenberger–Horne–Zeilinger (GHZ) state [11]

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0^{\otimes N}\rangle + |1^{\otimes N}\rangle). \tag{2}$$

Concerning for example the question of distillability, one may consider distillability of bipartite entanglement between pairs of particles or of true N-partite entanglement between a group of particles. In both cases, one may either ignore the remaining particles or allow them to assist the other parties in order to distil a MES. One may also consider partitions of the system, i.e. allowing some of the parties to act together and perform joint operations, and determine the distillability (and separability) properties with respect to this partition, which in turn provide information about the entanglement properties of the whole system. Each of the

situations just described is concerned with a different aspect of multiparticle entanglement, and will be discussed in more detail in the following.

From an experimental point of view, it is of particular interest to detect whether a N-particle state is distillable to a MES of N-particles. We will provide a simple, sufficient criterion which allows—without full knowledge of the density matrix—to detect true N-qubit entanglement. In addition, this criterion allows to detect different kinds of multiparticle entanglement as well. We also observe that there exist more kinds of multipartite entanglement then the obvious ones already mentioned previously (all possible combinations of maximally entangled l-partite states for different l). In particular, we consider bound entangled states, i.e. non-separable, non-distillable states and show that they can be activated under certain circumstances. We provide examples illustrating quite surprising effects related to bound entanglement and its activation.

This paper is organized as follows. We start in section 2 by discussing bipartite aspects of MPE, that is entanglement which is robust against disposal of particles. We discuss the necessary information which is required to detect these aspects of MPE. In section 3, we choose a different approach and concentrate on l-partite aspects of MPE. Using l-partite splittings of the system, we show how to completely determine the separability and distillability properties of a certain family of states, i.e. its MPE properties. Using these results, we provide a simple (sufficient) criterion to (experimentally) detect different kinds of MPE. We illustrate this method by applying it to two recent experiments, where MES of 3 (4) particles respectively were created. Finally, in section 4 we focus on an interesting phenomenon related to MPE, namely on bound entanglement and its activation. In particular, we show that bound entanglement can sometimes be activated by joint actions of some of the parties or alternatively with the help of a different kind of bound entanglement. We give several examples to illustrate these effects.

2. Entanglement which is robust against disposal of particles

In this section, we concentrate on bipartite aspects of multipartite entanglement, in particular on bipartite entanglement which is robust against disposal of particles. We consider N spatially separated parties $A_1, \ldots, A_N$, each possessing a qubit.

We say that two particles are (bipartite) entangled if their reduced density operator[1] is non-separable, i.e. the two particles share entanglement, independently of what happens to the remaining particles. When considering the reduced density operator of two parties, we deal with the situation where the information about all remaining particles is not accessible (or the remaining parties are not willing to cooperate). Such a definition is very suitable from a practical point of view, as there are certain multipartite scenarios where one is interested in entanglement properties of pairs of parties, which are independent of other parties. In addition, in certain experiments one may be faced with such a situation, e.g. when one of the particles escapes from a trap. The remaining particles should then be described by the reduced density operator. Note that in this sense, the GHZ state (2) contains no (bipartite) entanglement at all, as all reduced density operators are separable. However, the GHZ state can be regarded as MES of N particles in several other senses [12].

2.1. Entanglement molecules

In [6], it was shown that there exist N-particle states which are still entangled when tracing out *any* $(N-2)$ particles, i.e. there are states where all particles are entangled with all other

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[1] Given an N-partite state ρ, the reduced density operator ρ_{12} of parties A_1 and A_2 is defined as $\rho_{12} \equiv \mathrm{tr}_{3,\ldots,N}(\rho)$. The operator ρ_{12} is separable if it can be written as a convex combination of product states.

particles. In addition, it was shown there that there exist N-partite states ρ where one can choose for each of the $N(N-1)/2$ reduced density operators ρ_{kl} *independently* whether it should be separable or inseparable. This allows us to build general structures of N-particle states, which were called 'entanglement molecules' in [6].

The following family of N-qubit states includes all possible configurations of 'entanglement molecules' [6]. First we specify for each of the reduced density operators ρ_{kl} whether it should be distillable or not[2], i.e. whether entanglement between the parties A_k and A_l can be distilled—without the help of the remaining parties—or not. Let $I = \{k_1 l_1, \ldots, k_M l_M\}$ be the set of all those pairs where distillation should be possible, i.e. for $kl \in I$, we have that ρ_{kl} is distillable. We define the state

$$|\Psi_{ij}\rangle \equiv |\Psi^+\rangle_{ij} \otimes |0 \ldots 0\rangle_{\text{rest}} \tag{3}$$

that is, the particles A_i and A_j are in a MES, namely $|\Psi^+\rangle = 1/\sqrt{2}(|01\rangle + |10\rangle)$, and the remaining particles are disentangled from each other and from $A_i A_j$. The family of states

$$\rho_I = \frac{1}{M} \sum_{kl \in I} x_{kl} |\Psi_{kl}\rangle \langle \Psi_{kl}| \tag{4}$$

has the desired properties, which can be checked [6] by calculating the reduced density operators ρ_{kl} and using the partial transposition criterion. We have that $M \equiv \sum_{kl \in I} x_{kl}$ is a normalization factor. The bipartite aspects of multipartite entanglement were also analysed in [13].

2.2. Experimental detection

Given an N-qubit state ρ, how can we determine its (bipartite) entanglement properties? One possibility is to completely determine the N-partite density matrix of ρ. Given ρ, one can easily calculate all possible reduced density operators ρ_{kl} and determine the separability properties of each ρ_{kl}. Due to the fact that we deal with qubits, one can use the partial transposition criterion (see section 1.1) to determine for each of the reduced density operators ρ_{kl} whether it is separable or distillable. In the case that ρ_{kl} is inseparable, a MES shared by the parties A_k and A_l can be distilled.

However, it is rather difficult to completely determine the density matrix of N-qubit system, which is required in the procedure described above. Alternatively, one can concentrate from the very beginning on the properties of the reduced density operators ρ_{kl}, i.e. ignoring the remaining particles and just measuring the bipartite density operator ρ_{kl}. In this case, all $N(N-1)/2$ different reduced density operators have to be determined independently and can then be analysed using the partial transposition criterion.

Still, it might be too demanding to completely determine the density matrix of a two qubit system, which is necessary to completely determine the separability properties of this system. However, in order to *detect* entanglement in a two-qubit system, it suffices to show that the fidelity F, i.e. the overlap with an arbitrary MES, fulfils $F > 1/2$. Note however that this is a sufficient condition for inseparability (distillability), which is in general not necessary. So one can alternatively measure the overlap of each of the reduced density operators ρ_{kl} with a MES. Observing that for a given ρ_{kl}, $F > 1/2$ implies that out of ρ_{kl} a MES shared among A_k

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[2] Note that the inseparability of a certain reduced density operator ρ_{kl} already implies that a MES—such as $|\Phi^+\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle)$—shared between parties A_k and A_l can be distilled (when allowing for several copies of the state), even without the help of the remaining parties. This is due to the fact that for two qubit systems, inseparability is equivalent to distillability [10]. In fact, the remaining parties can by no means prevent parties A_k and A_l from distilling a MES.

and A_l can be distilled. However, when one finds $F \leqslant 1/2$, nothing can be concluded about the separability properties of ρ_{kl}.

Establishing the (bipartite) entanglement properties of a state ρ is however not the only possibility to determine the multipartite entanglement (MPE) properties of ρ. The bipartite entanglement properties, i.e. the properties of the reduced density operators ρ_{kl}, are only a certain aspect of the MPE properties of ρ. There are other aspects of MPE and alternative ways to detect the presence of MPE, which will be discussed in the next section.

3. *l*-partite aspects of multiparticle entanglement

In this section we first review the concepts of *l*-partite splittings, *l*-separability and distillability. These properties can be used to completely characterize the multiparticle entanglement properties of an arbitrary mixed state ρ [14]. We then review the properties of a family of *N*-qubit states ρ_N introduced in [2] and completely determine the entanglement properties of this family. Finally we show that these results can be used to determine entanglement properties of general states ρ without complete knowledge of the density matrix. In particular, it suffices to determine the diagonal matrix elements of ρ in a certain basis in order to establish sufficient conditions for the presence of multipartite entanglement. We provide a simple prescription to experimentally detect different kinds of multipartite entanglement. We apply the results to two recently performed experiments [15, 16] to illustrate the usefulness of our method.

3.1. Bipartite and l-partite splittings

Let us denote by $\mathcal{P}$ the set of all possible bipartite splittings of *N* parties into two groups. For example, for three parties $\mathcal{P}$ contains the splittings (A_1A_3)–(A_2), (A_2A_3)–(A_1), and (A_3)–(A_1A_2). We will denote these bipartite splittings by P_k, where $k = k_1 k_2 \ldots k_{N-1}$ is a chain of $N-1$ bits, such that $k_n = 0, 1$ if the *n*th party belongs to the same group as the last party or not. For example, for three parties the splittings (A_1A_3)–(A_2), (A_2A_3)–(A_1), and (A_3)–(A_1A_2) will be denoted by P_{01}, P_{10}, and P_{11}, respectively. We will denote by A the side of the splitting to which the party N belongs and by B the other side. In a similar way, one can consider *l*-partite splittings S_l, where the parties form exactly *l* groups. In the following, when we consider *l*-partite splittings, the parties in each of the *l* groups will be allowed to act together (i.e. to perform joint operations).

3.2. l-separability and distillability

Here we review the notion of separability and distillability in the case of multiparticle systems. We consider *N* parties, each holding a system with dimension d_i, i.e. $\mathcal{H} = \mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_N}$. We call ρ fully separable if it can be written as a convex combination of (unnormalized) product states, i.e.

$$\rho = \sum_i |a_i\rangle_{\text{party1}} \langle a_i| \otimes |b_i\rangle_{\text{party2}} \langle b_i| \otimes \ldots \otimes |n_i\rangle_{\text{party}N} \langle n_i|. \tag{5}$$

In the following, we will consider a system of *N* qubits, each held by one of the parties A_1, $A_2, \ldots, A_N$. In this case, $d_1 = d_2 = \ldots d_N = 2$. A state ρ is called *k*-separable with respect to a specific *k*-partite splitting iff it is fully separable in the sense that we consider ρ as a *k*-party system, i.e. as a state in $\mathcal{H} = \mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_k}$. In order to completely determine the separability properties of a state ρ, one should determine the separability properties of *all* possible *l*-partite splittings for all $l \leqslant N/2$. Based on this information, one can establish

an hierarchic classification of the entanglement properties of ρ (see [14] for details). It turns out that the separability properties of the different l-partite splittings for different l are not independent of each other, which strongly simplifies the classification and reduces the number of possible classes. In some cases we will deal with in the following, it even suffices to determine the biseparability properties of a state, i.e. to establish the separability properties of all possible bipartite splittings. This is due to the fact that the l-separability properties in this case are completely determined by the biseparability properties of ρ.

In a similar way, one can establish the distillability properties of a state ρ. Given a bipartite splitting P_k, a state ρ is called distillable with respect to the splitting P_k, if—out of N identical copies of ρ—the two groups A and B (which correspond to the two groups of the splitting) can create by means of local operations and classical communication a MES such as $|\Phi^+\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle)$, shared among A and B. Recall that the term 'local' in this case refers to local operation with respect to the groups A and B, but may involve joint operation on the particles within one group. In the case of distillability, it is not necessary to consider l-partite splitting and the possible creation of l-party GHZ states, as the creation of pairwise entanglement between any two out of l parties is a necessary and sufficient condition for the distillation of a l-partite GHZ state shared among those parties [15]. However, one may ask whether two subgroups—not containing all parties—are capable of distilling a MES with the help of the remaining parties. For a certain family of states, we will give necessary and sufficient conditions when this is possible.

3.3. Family of states ρ_N

Let us consider ρ_N, the family of N-qubit states introduced in [2]. We have that $\rho \in \rho_N$ if it can be written as

$$\rho = \sum_{\sigma=\pm} \lambda_0^\sigma |\Psi_0^\sigma\rangle\langle\Psi_0^\sigma| + \sum_{k\neq 0} \lambda_k(|\Psi_k^+\rangle\langle\Psi_k^+| + |\Psi_k^-\rangle\langle\Psi_k^-|) \qquad (6)$$

where

$$|\Psi_k^\pm\rangle \equiv \frac{1}{\sqrt{2}}(|k_1 k_2 \ldots k_{N-1} 0\rangle \pm |\bar{k}_1 \bar{k}_2 \ldots \bar{k}_{N-1} 1\rangle) \qquad (7)$$

are GHZ-like states with $k = k_1 k_2 \ldots k_{N-1}$ being a chain of $N-1$ bits, and $\bar{k}_i = 0, 1$ if $k_i = 1, 0$, respectively. We have that ρ_N is parametrized by 2^{N-1} independent real numbers. The labelling is chosen such that $\Delta \equiv \lambda_0^+ - \lambda_0^- \geqslant 0$. As we will see below, both the separability and distillability properties of the states belonging to this family are completely determined by the coefficients

$$s_k \equiv \begin{cases} 1 & \text{if } \lambda_k < \Delta/2 \\ 0 & \text{if } \lambda_k \geqslant \Delta/2. \end{cases} \qquad (8)$$

Let us emphasize that the notation used for the states of this family parallels the one used to denote the partitions P_k, i.e. there is a one-to-one correspondence between P_k and s_k. Note that there are no restrictions on the values of these coefficients; that is, for any choice of $\{s_k\}$ there always exists a state $\rho \in \rho_N$ with these values. We will now summarize the properties of states belonging to the family (6) [14, 4].

(i) *Depolarization.* An arbitrary state ρ can be depolarized to the standard form (6) by a sequence of N-local operations while keeping the values of $\lambda_0^\pm \equiv \langle\Psi_0^\pm|\rho|\Psi_0^\pm\rangle$ and $2\lambda_j \equiv \langle\Psi_j^+|\rho|\Psi_j^+\rangle + \langle\Psi_j^-|\rho|\Psi_j^-\rangle$ unchanged [14].

(ii) *Separability.* For any bipartite splitting $P_k \in \mathcal{P}$, and $\rho \in \rho_N$ we have $\rho^{T_A} \geqslant 0 \Leftrightarrow s_k = 0 \Leftrightarrow \rho$ is separable with respect to this splitting[3] [14]. More generally, $\rho \in \rho_N$ is *l*-separable with respect to a specific *l*-partite splitting S_l iff all bipartite splittings P_k which contain[4] S_l are separable (have $s_k = 0$).

(iii) *Distillability.* Let $\rho \in \rho_N$, $C = \{A_{i_1}, \ldots, A_{i_M}\}$ and $D = \{A_{j_1}, \ldots, A_{j_L}\}$ be two disjoint groups of M and L parties respectively, whereas the rest of the parties are separated. A MES between C and D can be distilled iff ρ is non-separable with respect to all those bipartite splittings P_k in which the groups C and D are located on different sides (i.e. all corresponding $s_k = 1$). It follows that ρ is distillable with respect to a bipartite splitting $P_k \Leftrightarrow s_k = 1$ [4].

Note that (ii) and (iii) *completely* determine the separability and distillability properties of an arbitrary state $\rho \in \rho_N$ and thus the multipartite entanglement properties of this state. We also have that (iii) already implies complete knowledge about the distillability of *k*-partite GHZ states, as the creation of pairwise entanglement between any two out of *k* parties is a necessary and sufficient condition for the distillation of a *k*-partite GHZ state shared among those parties [15].

3.4. Implications for experimental detection of multipartite entanglement

We have that (i)–(iii) together provide a simple criterion for the detection of multipartite entanglement for arbitrary mixed states ρ. From (i) it follows that any state ρ is *at least* as entangled as the depolarized version $\tilde{\rho} \in \rho_N$ of ρ. This is due to the fact that a sequence of local operations may destroy some entanglement, but cannot create any new kind of entanglement which was not present in the initial state. This already gives us a prescription to detect different kinds of multipartite entanglement of an arbitrary state ρ:

- Determine the following diagonal matrix elements of ρ:
$$
\begin{aligned}
\lambda_0^\pm &\equiv \langle \Psi_0^\pm | \rho | \Psi_0^\pm \rangle \\
2\lambda_j &\equiv \langle \Psi_j^+ | \rho | \Psi_j^+ \rangle + \langle \Psi_j^- | \rho | \Psi_j^- \rangle \\
&= \langle j0 | \rho | j0 \rangle + \langle \bar{j}1 | \rho | \bar{j}1 \rangle.
\end{aligned}
\tag{9}
$$

Note that determining $\lambda_0^\pm$ requires a measurement in an entangled basis (GHZ basis), while determining λ_j corresponds to a measurement in a product basis. Recall that $|j0\rangle = |j_1 j_2 \ldots j_{N-1} 0\rangle$ and $|\bar{j}0\rangle = |\bar{j}_1 \bar{j}_2 \ldots \bar{j}_{N-1} 1\rangle$ (see (7)). Equivalently, it suffices to determine all *diagonal* matrix elements of ρ in the standard basis plus one off-diagonal element, namely $|0\ldots0\rangle\langle1\ldots1|$.

- Calculate $\Delta = \lambda_0^+ - \lambda_0^- = 2\,\mathrm{Re}(\langle 0\ldots0 | \rho | 1\ldots1 \rangle)$ and determine the coefficients s_k given in (8). If at least one $s_k = 1$, we have that ρ is entangled.

- Use (i)–(iii) to determine the (minimal) entanglement properties of the state ρ. Note that obtaining $s_k = 0$ for a certain bipartite splitting P_k does not imply that ρ is separable with respect to this splitting. It might well be that ρ is inseparable (entangled) with respect to P_k, but the corresponding depolarized state $\tilde{\rho}$ is separable. However, obtaining $s_k = 1$ ensures that a certain kind of entanglement is present in the state ρ—namely that ρ is inseparable with respect to the bipartite splitting P_k. In particular, one can distil a GHZ state from ρ iff $s_k = 1 \; \forall k$.

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[3] ρ^{TA} denotes the partial transposition with respect to the parties A. For the definition of partial transposition in multiparticle systems see [8, 14]. The relation between subsystem A and P_k is given in section 3.1.

[4] A *l*-partite splitting S_l is contained in a *k*-partite splitting P_k iff P_k can be obtained from S_l by joining some of the parties of S_l.

3.5. Application to recent experiments

Let us apply this method to two recent experiments performed by Sacket *et al* [15] and Rauschenbeutel *et al.* [16].

In [15], the creation of an (mixed) entangled state ρ of four ions, whose overlap with the GHZ state $|\Psi_0^+\rangle$ is $F = 0.57 \pm 0.02$, was reported. It was argued that it is sufficient to obtain $F > 1/2$ in order to be sure that the state is 4-partite entangled. This sufficient criterion is however—in some cases—much too demanding and can be relaxed using the results presented in this work. Imagine for example that the 4-qubit state ρ is of the form

$$\rho(x) = x|\Psi_0^+\rangle\langle\Psi_0^+| + \frac{1-x}{16}\,\mathbb{1}_4 \qquad\qquad (10)$$

This is clearly a special case of the state ρ_4 with $\lambda_0^- = \lambda_j = \frac{1-x}{16}, \lambda_0^+ = x + \frac{1-x}{16}$ and thus $\Delta = x$. Using (ii) and (iii), we can state that $\rho(x)$ is fully non-separable and distillable to a four party GHZ state iff $x > 1/9$, which corresponds to $F > 1/6$ [15]. Note that the bound $F > 1/2$—which is independent of the number of parties N—corresponds to a worst-case scenario, where it is assumed that $\lambda_0^+ = F$ and the remaining weight is distributed on λ_0^- and one specific λ_k. In this case, we have for $F > 1/2$ that $\Delta > 2\lambda_k \,\forall k$. If the remaining weight $(1-F)$ is however distributed on λ_0^- and more than one λ_k, it automatically follows that $\Delta > 2\lambda_k \,\forall k$ is already fulfilled for all $\lambda_0^+ \equiv F > F_0$, where $F_0 < 1/2$. The weakest bound on the fidelity F can be obtained by assuming that the state $\tilde{\rho}_N$ is of the following form: $\lambda_0^+ = F, \lambda_0^- = 0$ and $2\lambda_k = (1 - F)/(2^N - 2)$. This ensures that $\tilde{\rho}_N$ has $\Delta > 2\lambda_k \,\forall k$ and is thus distillable to an N-party GHZ state iff $F > 1/(2^N - 1)$. For $N = 4$, we obtain $F > 1/15$. We thus have that additional knowledge of the shape of the state may relax the necessary conditions to ensure that a state is entangled.

Let us now focus on the specific experiment [15] and apply these observations. Unfortunately, the published experimental data are not sufficient to determine all coefficients λ_k. However, one can easily determine

$$\begin{aligned}\lambda_0^\pm &= 1/2(\langle0000|\rho|0000\rangle + \langle1111|\rho|1111\rangle) \\ &\quad \pm \mathrm{Re}(\langle0000|\rho|1111\rangle) = 0.35 \pm 0.215(\pm0.02),\end{aligned} \qquad (11)$$

from which follows that $\Delta = 0.43(\pm0.02)$. In addition, one can also bound the other coefficients λ_k and find[5]

$$\begin{aligned}0 \leqslant 2\lambda_k &\leqslant 0.2(\pm0.04) & &\text{iff } k \in \{001, 010, 100, 111\} \\ 0 \leqslant 2\lambda_k &\leqslant 0.1(\pm0.02) & &\text{iff } k \in \{011, 101, 110\}.\end{aligned} \qquad (12)$$

We thus have that $\Delta > 2\lambda_k \,\forall k$ as expected. Note however that a fidelity $F < 1/2$ would have been sufficient to ensure that the produced state is truly 4-partite entangled. Assume, for

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[5] In [15], only the probabilities p_l to find l ions in state $|0\rangle$ and $N-l$ ions in state $|1\rangle$ are given. For $l = 1$, there are four states which might contribute to p_1, namely $|\phi_i^1\rangle \in \{|0111\rangle, |1011\rangle, |1101\rangle, |1110\rangle\}$. From the measured data we know that $p_1 = P_1\rho P_1 = \mathrm{tr}(\sum d_{ij}^1 |\phi_i^1\rangle\langle\phi_j^1|)$, where $P_1 = \sum |\phi_j^1\rangle\langle\phi_j^1|$ and we assumed that d_{ij}^1 are the (unknown) coefficients of the density matrix ρ in the basis $|\phi_i^1\rangle$. Similarly, for $l = 3$, the four states $|\phi_i^3\rangle \in \{|0001\rangle, |0010\rangle, |0100\rangle, |1000\rangle\}$ can contribute to p_3 and $\mathrm{tr}(\sum d_{ij}^3|\phi_i^3\rangle\langle\phi_j^3|) = p_3$. For $l = 2$, there are six states which might contribute to p_2, namely $|\phi_i^2\rangle \in \{|0011\rangle, |0101\rangle, |0110\rangle, |1001\rangle, |1010\rangle, |1100\rangle\}$ and again $\mathrm{tr}(\sum d_{ij}^3|\phi_i^3\rangle\langle\phi_j^3|) = p_3$. Note that the coefficients d_{ij}^l are not specified by the provided data, but the diagonal coefficients d_{ii}^l are bound from above by p_l. We have e.g. that $2\lambda_{001} = d_{11}^1 + d_{22}^3$. Since we might get contributions to λ_{001} from p_1 and p_3, we conclude that $0 \leqslant 2\lambda_{001} \leqslant p_1 + p_3$. A similar argument holds e.g. for $2\lambda_{110} = d_{66}^2 + d_{11}^2$, but in this case we only get contributions from p_2. We thus conclude that $0 \leqslant 2\lambda_{110} \leqslant p_2$. The same line of argument can be used to obtain bounds for all λ_k. In the experiment, the values of p_j were determined to be $p_1 = p_2 = p_3 = 0.10 \pm 0.02$.

example, that white noise is added to the experimentally produced state, i.e. $\tilde{\rho} = x\rho + (1 - x)/16\mathbb{1}_4$. Using the bounds on λ_k just derived, we find that $\tilde{\rho}$ remains truly 4-partite entangled for $x > 0.3521$, which corresponds to a fidelity of $F > 0.3926$, significantly below $1/2$.

Our method should thus simplify the task of detecting an entangled state of a larger number of particles ($N > 4$), as it relaxes the necessary conditions for the detection of true N-partite entanglement. Note that it would be highly desirable to measure all diagonal coefficients in the standard basis independently rather than projections into subspaces P_j with j particles in $|0\rangle$ and $N-j$ particles in $|1\rangle$ as done in [16]. Doing so, one could determine the coefficients λ_k directly and would not have to use a 'worst-case scenario' in order to establish bounds on λ_k as we did here (see footnote 5). In addition, different kinds of entanglement which do not correspond to N-party GHZ entanglement can be detected as well. In the next section, we show that states showing these different kinds of entanglement may also be interesting to produce, as they provide examples of surprising effects such as the activation of bound entanglement.

One may also adopt this method to other experiments, such as the one performed by Rauschenbeutel *et al* [16], where a maximally entangled state of three spin-$\frac{1}{2}$ systems (two atoms plus one cavity mode) was created. Let us first adopt the notation used in [16] to the one used throughout this paper: $|+_j\rangle = |1\rangle, |-_j\rangle = -|0\rangle$, where e.g. $|+_1\rangle = |e_1\rangle, |+_2\rangle = (|g_2\rangle + |i_2\rangle)/\sqrt{2}$ and $|+_C\rangle = |g_3\rangle$ (see equation (3) and below in [16]). In addition, we make a basis change $|0\rangle \rightarrow -|1\rangle$ and $|1\rangle \rightarrow |0\rangle$ in party 3. It follows that the longitudinal correlations given in figure 3 of [16] correspond to the diagonal matrix elements of ρ in the basis (from left to right) $\{|011\rangle, |010\rangle, |001\rangle, |000\rangle, |111\rangle, |110\rangle, |101\rangle, |100\rangle\}$. From this we can determine

$$2\lambda_{01} = 0.14(\pm 0.04)$$
$$2\lambda_{10} = 0.155(\pm 0.04) \tag{13}$$
$$2\lambda_{11} = 0.128(\pm 0.04)$$

From the transverse correlations we find[6]

$$\Delta = 2\text{Re}(\langle 000|\rho|111\rangle = 2V_\perp = 0.28(\pm 0.04). \tag{14}$$

Thus we have that $\Delta > 2\lambda_k \ \forall k$ and we can conclude that the experimentally detected state ρ is in fact distillable to a 3-party GHZ state. Note that in [16], it was necessary to take known detection errors into account in order to obtain $F > 0.5$. Here we can state that even without taking these errors into account, the state ρ is true tripartite entangled, although its fidelity $F = 0.43 < 1/2$.

3.6. Bound entanglement and its activation

Let us now consider N spatially separated parties, $A_1, \ldots, A_N$, who share M identical copies of an N-qubit state ρ, where M can be as large as we wish. This ensures that the parties can use distillation protocols [17] in order to obtain MES between some of them. In the case that this is possible, we say that the state ρ is distillable (with respect to the specific parties which obtain the MES). If no MES shared between any two of the parties can be distilled and in addition the state ρ is not fully separable (i.e. entangled), we say that ρ is bound entangled (BE).

3.7. Activating bound entanglement by joint actions

Given a bound entangled state (BES), in some cases it is possible to activate the bound entanglement. We say that a BES can be activated if it becomes distillable once some of the

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[6] The quantity $V_\perp$ measured in the experiment also depends on other off-diagonal elements.

parties join and form groups to act together. Note that instead of allowing some parties to join we could have allowed them to share some extra MES. In that case we would have the same situation given the fact that separated parties sharing MES can perform any arbitrary joint operation by simply teleporting [18] back and forth the states of their particles.

The first example of this kind was given in [2]. There it was shown that given a certain BES shared by three parties, providing some extra bipartite entanglement between A_1 and A_2 enables the three parties to create a tripartite *GHZ* state.

In [3], Smolin presented another example of this kind involving four parties. This example has the additional feature that only a single copy of a BES ρ is required in order to distill a MES shared by two of the parties (say A_1 and A_2) once the other two parties (say A_3 and A_4) are allowed to act together and perform joint operations.

Using states of the form (6), several examples showing the activation of different kinds of BE by joint actions of some of the parties were provided in [4]. In addition, a systematic method for the construction of different kinds of activatable BES was provided there. Let us review some of the examples given in [4]:

Example 1. The state ρ_I becomes distillable iff the parties form two groups with exactly j and $N-j$ members, respectively. Furthermore, it does not matter which of the parties join in each group, but only the number of members. For example, if $N = 8$ and $j = 3$, we have that ρ_I is distillable if exactly three and five parties join, but remains undistillable when the parties form two groups with 1–7, 2–6, 4–4 members, or if they form more than two groups. In particular, ρ_I is not distillable if the parties remain separated from each other, which corresponds to having eight groups. We can take as state ρ_I one from the family ρ_N which has $s_k = 1$ iff the number of ones in k is j or $(N-j)$ and $s_k = 0$ otherwise (this means that all bipartite splittings which contain exactly j members in one group are distillable, and all others are separable).

Example 2. The state ρ_{II} becomes distillable iff the parties form two groups, where the first group contains a *specific* set of L parties $A = \{A_{k_1}, \ldots A_{k_L}\}$, and the second group contains the remaining parties. For all other configurations in groups ρ_{II} remains undistillable. For example, we have for $N = 5$ and $A = \{A_1, A_3, A_5\}$ that ρ_{II} is distillable iff the parties form two groups, $(A_1A_3A_5)$–(A_2A_4), and not distillable otherwise. We can take $\rho_{II} \in \rho_N$ such that $s_k = 1$ only for one specific P_k. For $N = 5$, choosing $s_{0101} = 1$ ensures that ρ_{II} is inseparable and thus distillable with respect to the bipartite splitting $(A_1A_3A_5)$–(A_2A_4) and separable (and thus undistillable) otherwise.

Example 3. ρ_{III} is a BES of $N = 4$ parties for which, once the parties (A_3A_4) form a group, a GHZ-like state can be distilled among A_1, A_2, and the group (A_3A_4), whereas it is undistillable whenever any other parties but (A_3A_4) are joint. We choose $\rho_{III} \in \rho_4$ such that it is inseparable with respect to the bipartite splittings (A_1A_2)–(A_3A_4), (A_1)–$(A_2A_3A_4)$ and (A_2)–$(A_1A_3A_4)$ and separable with respect to all other bipartite splittings.

The described activation effects can be understood using (ii) and (iii) of section 3.3, together with the fact that when joining some of the parties, one may change the separability properties of certain bipartite splittings P_k from separable to inseparable (see [4] for details).

We conclude that the experimental creation of non-maximally entangled N-partite states (not all $s_k = 1$) might be of interest as well, as those states can have quite surprising properties. Note however that, in this case, it is essential that the produced states are of the form (6), which can be accomplished by physically implementing the depolarization procedure described in [14].

3.8. Activating bound entanglement with bound entanglement

Let us now consider the situation where the N parties possess different kinds of BES, ρ_1, $\rho_2, \ldots \rho_L$, but this time remain spatially separated from each other. The parties again possess several copies of each of the states, i.e. $\rho_i^{\otimes M_i}$. By definition, it is clear that $\rho_i^{\otimes M_i}$ is not distillable for all i, i.e. the parties cannot create a MES if they have access to only one kind of BES.

However if the parties have access to all different kinds of BES, i.e they share the state

$$\rho' = \otimes_{i=1}^{L} \rho_i^{\otimes M_i} \tag{15}$$

we will give examples where they can distil a MES between some of the parties or even a GHZ state shared among all the parties. This effect, namely that the tensor product of two BES, is no longer necessarily a BES, was discovered by Shor *et al* [5] and was termed 'superactivation'. We shall refer to this as activation of bound entanglement with bound entanglement.

Let us investigate the simplest example of a tripartite system, $N = 3$. We consider a state ρ_1 which is inseparable with respect to the bipartite splitting A–BC and separable with respect to the splittings B–AC and C–AB. As shown in [4], such a state is BE (a necessary condition for distillation of a MES shared between any two of the three parties is that at least two of the bipartite splittings have to be inseparable). Now consider states ρ_2 and ρ_3 which are created from the state ρ_1 by cyclic permutations of the parties, i.e. ρ_2 (ρ_3) is inseparable with respect to the splitting B–AC (C–AB) respectively. For a particular choice of the states ρ_1, ρ_2, ρ_3, the parties can create—once they have access to all three kinds of states—a state $\tilde{\rho}$ which is inseparable with respect to all three bipartite splittings and which is in addition distillable to a GHZ state. In fact, they just have to pick randomly one of the three states ρ_1, ρ_2, ρ_3 (this can be accomplished via classical communication only), i.e.

$$\tilde{\rho} = \frac{1}{3}(\rho_1 + \rho_2 + \rho_3). \tag{16}$$

To be specific, choosing ρ_1 within the family of states (6), $N = 3$ with the following coefficients

$$\lambda_0^+ = \frac{1}{3} \qquad \lambda_0^- = \lambda_2 = 0 \qquad \lambda_1 = \lambda_3 = \frac{1}{6} \tag{17}$$

ensures (i) that ρ_1 and ρ_2, ρ_3 (created by cyclic permutations of the parties) are BE with separability properties with respect to the bipartite splittings as announced above. (ii) $\tilde{\rho}$ defined in (16) is again of the form (6) with coefficients

$$\tilde{\lambda}_0^+ = \frac{1}{3} \qquad \tilde{\lambda}_0^- = 0 \qquad \tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda}_3 = \frac{1}{9} \tag{18}$$

and is inseparable with respect to all bipartite splittings (since $\Delta > 2\lambda_k$) and hence distillable to a GHZ state.

It is now straightforward to extend these ideas to more parties and to a more general setup. Therefore we consider a subfamily of N-qubit states of the form (6). We denote by S all those bipartite splittings P_k for which the state ρ is inseparable (the corresponding $s_k = 1$). For all other bipartite splittings $P_k \notin S$, ρ is separable (the corresponding $s_k = 0$). Let the number of separable bipartite splittings be $s > 0$. We define $\Delta \equiv 1/(s + 1)$. The subfamily is defined by

the following choice of parameters:

$$
\begin{aligned}
\lambda_0^+ &= \Delta & \lambda_0^- &= 0; \\
\lambda_k &= 0 & &\text{iff } P_k \in S \\
\lambda_k &= \frac{\Delta}{2} & &\text{iff } P_k \notin S.
\end{aligned}
\tag{19}
$$

In general, we can announce the following.

Theorem 1. *Given L different kinds of BES $\rho_1, \ldots, \rho_L$ of the form (19), where S_j denotes all bipartite splittings with respect to which ρ_j is inseparable, one can create a state $\tilde{\rho}$ which is inseparable with respect to all those bipartite splittings where at least one of the states ρ_j was inseparable, i.e. $\tilde{S} = \cup S_j$.*

Proof. We define

$$
\tilde{\rho} = \frac{1}{L} \sum_{j=1}^{L} \rho_j,
\tag{20}
$$

i.e. we pick randomly one of the states ρ_j (which can be accomplished by classical communication) and show that $\tilde{\rho}$ has the desired properties. We have that $\tilde{\rho}$ is again of the form (6) and the coefficients $\tilde{\lambda}_k$ are given by the average of the coefficients $\lambda_{k,j}$ of the states ρ_j. We have to show that (i) $\tilde{\Delta} > 2\tilde{\lambda}_k$ iff $P_k \in \tilde{S} \equiv \cup S_j$ and (ii) $\tilde{\Delta} \leqslant 2\tilde{\lambda}_k$ iff $P_k \notin \tilde{S}$. We have that $\tilde{\Delta} = 1/L \sum_{j=1}^{L} \Delta_j$. In case (i), we have that at least one of the states ρ_j is inseparable with respect to the splitting P_k. We assume without loss of generality that it is only one, namely ρ_1, and thus $\lambda_{k,1} = 0$ (the argument is exactly the same if more than one of the states ρ_j are inseparable with respect to P_k). In this case we obtain for the corresponding $\tilde{\lambda}_k = 1/L \sum_{j=2}^{L} \Delta_j/2$. Note that the sum runs from $j=2$ to L, which ensures that (i) is fulfilled—since $(\tilde{\Delta} - 2\tilde{\lambda}_k) = \Delta_1/L > 0$. In the case of (ii), i.e. $P_k \notin S_j \ \forall j$, we find $\tilde{\lambda} = 1/L \sum_{j=1}^{L} \Delta_j/2$ and (ii) is fulfilled, which finishes the proof of our statement. $\square$

Given this theorem, it is now very easy to construct several examples which show the activation of bound entanglement with bound entanglement.

Example 1. We consider N parties and assume that N is even. We have $N/2$ different BE states $\{\rho_k\}$, $k = 1, 2, \ldots, N/2$. If the parties have access to any $(N/2 - 1)$ (or fewer) different kinds of BE states ρ_k, they cannot distil any entanglement. However, once the parties have access to all kinds of BE states ρ_k, they can create a state $\tilde{\rho}$ which is inseparable with respect to all bipartite splittings and thus distillable to an N-party GHZ state. The following choice of states has the announced properties: the state ρ_k is of the form (19) and is inseparable with respect to all bipartite splittings which contain exactly k parties on one side and $N-k$ parties on the other side and separable with respect to all other bipartite splittings. This ensures that all state ρ_k are BE [4] and—according to theorem 1—the parties can create a state $\tilde{\rho}$ which is distillable to a GHZ state once they have access to all $N/2$ different states ρ_k. If the access is limited to $(N/2 - 1)$ or fewer different kinds of BE states ρ_k, one can easily check using (iii) of section 3.1 that no entanglement can be distilled.

Example 2. In this example, we consider N different BE states $\{\rho_l\}$, $l = 1, 2, \ldots, N$. Here, the state ρ_N serves as a 'key-state', as, on one hand, access to ρ_N together with access to the state ρ_l enables the parties A_l and A_N to distil a MES. On the other hand, access to all states ρ_l except ρ_N does not allow the parties to distil any entanglement at all. If in addition also ρ_N is accessible, a GHZ state shared by all the parties can be distilled (as party A_N can create a

MES shared with any party A_l). Such a situation can be established by the following choice of states: for $l \neq N$, the state ρ_l is of the form (19) and is inseparable with respect to all bipartite splittings which have parties A_l and A_N on different sides, except the splittings A_l–rest and A_N–rest which as well as all the other splittings are separable. The state ρ_N is also of the form (19) and is inseparable with respect to all splittings where exactly one particle is on one side and $N - 1$ particles are on the other side. All states ρ_l are BE, which can be checked using (ii) and (iii) of section 3.3. Applying theorem 1, it is easy to observe the described activation effect.

Note that the activation of BE by joint actions may be combined with the activation of BE with BE. This opens a huge variety of different examples, which can all be constructed using the results of [4] together with theorem 1 and states of the form (19).

4. Summary

We discussed several aspects of multipartite entanglement and its experimental detection. First we focused on bipartite aspects of MPE, which can be determined by investigating the bipartite reduced density operators of the multipartite systems. We then used l-partite splittings to establish the l-separability and distillability properties of a multipartite density operator ρ. For a certain family of states, we completely determined the separability and distillability properties using bipartite splittings only. Using this, we provided a simple method to determine whether a mixed state ρ is multipartite entangled, and in addition to detect which kind of entanglement is present. We illustrated this method by revisiting two recent experiments. Finally, we focused on bound entangled states and the activation of BE. We showed that BE can be activated by joint actions of the parties or with the help of a different kind of BE itself.

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